# THE EFFECT OF NONSTEADINESS ON THE SHEARING STRESS AND VELOCITY PROFILE OF AN OPEN STREAM OF FLUID

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Recently there has been growing interest in the hydraulic resistance and the kinematics of nonsteady flow.

One of the first attempts to evaluate the effect of nonsteadiness on the velocity structure of a stream was the investigation by V. V. Vedernikov and V. A. Arkhangel'skii [1].

The velocity structure of nonsteady flow was studied in detail by G. F. Fedorov. He, in particular, established [2], that the translational velocity at the free surface may turn out to be less than within the stream. An explanation of this is given below.

Paper [3] contains an investigation of the effect of nonsteadiness on the shearing stress. In it the following assumptions were made: (1) the velocity curves for uniform and nonsteady flow are similar, (2) the shearing stresses for uniform and nonsteady flow are the same at the bottom of the stream.

These assumptions are not made here. The shearing stress and velocity curves for the nonsteady motion of an open plane-parallel turbulent stream of fluid were investigated in an approximate way within the framework of the assumptions made in [4].

**§1. Frictional stress.** The flow of an open plane-parallel turbulent stream of fluid is considered for large Reynolds numbers and a small transverse component of acceleration. We then have the equations [4]

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=-g\left(\frac{\partial h}{\partial x}\cos\vartheta-\sin\vartheta\right)+\frac{1}{\rho}\frac{\partial\tau}{\partial y},$$

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0, \qquad (1.1)$$

and the boundary conditions

$$\partial h/\partial t+u\,\partial h/\partial x=v,\qquad \tau=0\quad\text{for}\quad y=h;$$

$$u=0,\qquad v=0\qquad\text{for}\quad y=0. \qquad (1.2)$$

Here t is time, x and y are rectangular Cartesian coordinates, with the x-axis in the direction of the rectilinear bottom of the stream, u and v are velocity components in the direction of the x- and y-axes, respectively, h is the depth, $\tau$ is the friction stress, $\rho$ is the density, g is the acceleration of gravity, $\vartheta$ is the angle of inclination of the bottom to the horizontal (taken to be positive). To complete system (1.1), additional considerations are necessary, such as the relations of the semiempirical theory of turbulence which are given in [4].

We now represent the ratio of the friction stress $\tau$ at some point in the stream to the friction stress $\tau_0$ at the bottom of the stream in the form of a polynomial of degree $\eta = y/h$. The coefficients of the polynomial

$$\tau/\tau_0=b_0+b_1\eta+b_2\eta^2+\ldots+b_n\eta^n\qquad(0\leqslant\eta\leqslant 1)$$

can be determined from (1.1) and (1.2). Here, as in [4], we confine ourselves to the simplest, one-parameter case, which corresponds to a polynomial with three terms; further increase in the number of terms in the polynomial introduces new nonsteadiness parameters.

To determine the coefficients $b_0$, $b_1$, and $b_2$ we start from the conditions:

(a) at the bottom where $\eta = 0$

$$\tau/\tau_0=1,$$

(b) at the bottom where $\eta = 0$

$$\frac{\partial\tau}{\partial y}=\rho g\left(\frac{\partial h}{\partial x}\cos\vartheta-\sin\vartheta\right),$$

$$\text{or}\quad\frac{\partial}{\partial\eta}\frac{\tau}{\tau_0}=\frac{\rho g h}{\tau_0}\left(\frac{\partial h}{\partial x}\cos\vartheta-\sin\vartheta\right)=A$$

(c) at the free surface where $\eta = 1$

$$\tau/\tau_0=0.$$

We then have

$$\tau/\tau_0=1+A\eta-(1+A)\eta^2\quad(0\leqslant\eta\leqslant 1). \qquad (1.3)$$

In the case of uniform steady motion $\partial u/\partial t = 0$ and $\partial u/\partial x = 0$, and from the second equation of (1.1) we have $v = 0$. Integration of the first equation of (1.1) with respect to y from 0 to h yields $A = -1$ in this case.

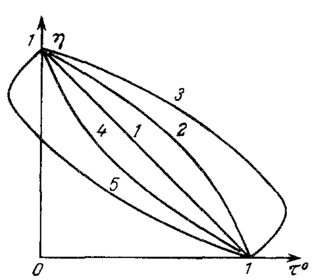


Fig. 1

Thus $\tau/\tau_0 = 1 - \eta$ in the case of uniform steady motion.

We now integrate the first equation of (1.1) with respect to y from 0 to h by means of the second equation of (1.1) and conditions (1.2). We then have

$$\frac{\partial}{\partial t}\int_0^h u\,dy+\frac{\partial}{\partial x}\int_0^h u^2dy=-gh\left(\frac{\partial h}{\partial x}\cos\vartheta-\sin\vartheta\right)-\frac{\tau_0}{\rho}.$$

Introducing the average velocity w and remembering that $u = w + \Delta u$, we obtain

$$A=-1+\delta,\qquad \delta=-\frac{1}{u_*^2}\left(\frac{\partial}{\partial t}wh+\frac{\partial}{\partial x}\alpha_1w^2h\right)$$

$$\left(\alpha_1=1+\frac{1}{w^2h}\int_0^h\Delta u^2dy>1,\quad u_*^2=\frac{\tau_0}{\rho}\right).$$

It is now clear that deviations of parameter A from the value $-1$ are due to the nonsteady nature of flow. Clearly, for accelerated motion $\delta < 0$ or $A < -1$.

We consider (1.3) for some values of x and t (parameters). We have from (1.3)

$$\tau^{\circ\prime}=d\tau^\circ/d\eta=A-2(1+A)\eta,$$

$$\tau^{\circ\prime\prime}=d^2\tau^\circ/d\eta^2=-2(1+A)\quad(\tau^\circ=\tau/\tau_0). \qquad (1.4)$$

1.1. Let $\delta < 0$ (accelerated flow). In this case, in accordance with (1.4), we obtain

$$\tau^{\circ\prime}=-1+\delta(1-2\eta),\quad \tau^{\circ\prime\prime}=-2\delta>0.$$

(1) The case of $-1 < \delta < 0$. Here $\tau^{\circ\prime} < 0$ and $\tau^{\circ\prime\prime} > 0$. Consequently, the value of $\tau^\circ(\eta)$ decreases monotonically as $\eta$ increases, and the curve for $\tau^\circ(\eta)$ is convex downward. It follows from this that $\tau^\circ(\eta) < \tau_C^\circ(\eta)$ for all values of $\eta$ in the interval $0 < \eta < 1$, where $\tau_C^\circ(\eta)$ is $\tau^\circ(\eta)$ for uniform motion ($\delta = 0$).

(2) The case of $\delta < -1$. Here $\tau^{\circ\prime} < 0$ for $0 \leq \eta < \eta^*$ ($\eta^* = (1 - 1/\delta)/2$), $\tau^{\circ\prime} > 0$ for $\eta^* < \eta \leq 1$, and $\tau^{\circ\prime} = 0$ at the point $\eta = \eta^*$,

$\tau^{\circ\prime\prime} > 0$, i.e., the function $\tau^\circ(\eta)$ decreases monotonically from unity for $\eta = 0$, and reaches a minimum for $\eta = \eta^*$. It subsequently increases monotonically to 0 for $\eta = 1$, and the curve for $\tau^\circ(\eta)$ is convex downward.

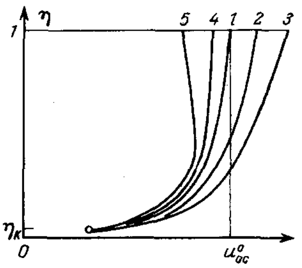


Fig. 2

1.2. Let $\delta > 0$ (retarded flow). The following conclusions can then be arrived at in the same way.

(1) For $0 < \delta < 1$ the value of $\tau^\circ(\eta)$ decreases monotonically as $\eta$ increases, and the curve for $\tau^\circ(\eta)$ is convex upward ($\tau^\circ(\eta) > \tau_c^\circ(\eta)$ for all values of $\eta$ in the interval $0 < \eta < 1$).

(2) For $\delta > 1$ the function $\tau^\circ(\eta)$ increases monotonically from unity for $\eta = 0$, and reaches a maximum for $\eta = \eta^*$. It then decreases monotonically to 0 for $\eta = 1$.

The distribution of shearing stress $\tau^\circ(\eta)$ is given in Fig. 1 where the curves correspond to the following values of $\delta$:

$$\overset{5}{\delta < -1}, \quad \overset{4}{-1 < \delta < 0}, \quad \overset{1}{\delta = 0}, \quad \overset{2}{0 < \delta < 1}, \quad \overset{3}{1 < \delta}.$$

**§2. Velocity profile.** According to [4], the velocity profile corresponding to the approximation of frictional stress, given by (1.3), satisfies the equation

$$\partial u^\circ/\partial\eta = (1 + A) + 1/\eta \qquad (u^\circ = \alpha u/u_*, \ u_0^\circ = \alpha u_0/u_*), \tag{2.1}$$

where $u^\circ = u_0^\circ$ for $\eta = 1$ is taken as the boundary condition. Here $\alpha$ is the first turbulence constant, and $u_0 = u$ for $\eta = 1$.

We note that, close to the bottom, viscous friction, which was not allowed for in (2.1), becomes important. Accordingly, (2.1) does not yield $u^\circ = 0$ right at the bottom as it should, but rather $u^\circ = -\infty$. Equation (2.1) is analyzed below in the interval $0 \le \eta \le 1$. The results of the analysis are valid, of course, only in the region of values of $\eta$ which excludes the region in the immediate vicinity of the bottom.

As in §1, we consider (2.1) for fixed values of x and t. Then we have from (2.1)

$$u^{\circ\prime\prime} = d^2u^\circ / d\eta^2 = -1/\eta^2 < 0. \tag{2.2}$$

2.1. Let $\delta < 0$ (accelerated flow). Then, in accordance with (2.1) and (2.2), we obtain $u^{\circ\prime} = du^\circ/d\eta = \delta + 1/\eta$ and $u^{\circ\prime\prime} < 0$. The following conclusions can then be drawn without difficulty.

(1) For $-1 < \delta < 0$ the value of $u^\circ(\eta)$ increases monotonically as $\eta$ increases, and the curve for $u^\circ(\eta)$ is convex upward.

(2) For $-\infty < \delta < -1$ the function $u^\circ(\eta)$ increases monotonically as $\eta$ increases, reaches a maximum at the point $\eta = -1/\delta$, and then decreases monotonically.

2.2. Let $\delta > 0$ (retarded motion). Then $u^{\circ\prime} > 0$ and $u^{\circ\prime\prime} < 0$. Consequently, the function $u^\circ(\eta)$ increases monotonically as $\eta$ increases, for all values of $\eta$, and the curve for $u^\circ(\eta)$ is convex upward.

Thus, we have $u_+^{\circ\prime} < u_c^{\circ\prime} < u_-^{\circ\prime}$ and $u_+^{\circ\prime\prime} = u_c^{\circ\prime\prime} = u_-^{\circ\prime\prime} < 0$ for any $\eta$. Here the subscripts +, c, and − denote accelerated, uniform, and retarded flow, respectively.

Treating the phenomenon as in [4], we set $u(\eta_k) = \beta u_*$, where $\eta_k = k/h$, $\beta$ is the second turbulence constant, and k is the mean height at which the roughness projections make themselves felt. The velocity profile $u^\circ(\eta)$ is given for the same depth in Fig. 2 (the symbols are the same as those in Fig. 1). It is clear from this, first, that we can speak about the similarity of velocity profiles for nonsteady and uniform flow only for the case in which $|\delta| < 1$, and second, that $u_+^\circ < u_c^\circ < u_-^\circ$ for all values of $\eta$ in the interval $\eta_k < \eta \le 1$, in particular

$$u_{0+}^\circ < u_{0c}^\circ < u_{0-}^\circ. \tag{2.3}$$

Inequalities (2.3) can also be obtained directly from the expression for the velocity profile resulting from (2.1). It follows from (2.3) that

$$u_{0+} / u_{*+} < u_{0c} / u_{*c} < u_{0-} / u_{*-}. \tag{2.4}$$

Inequalities (2.4) mean that for the same longitudinal velocity at the free surface and for the same depth, the fictional stress at the bottom is greater for accelerated motion than for uniform motion, and less for retarded motion than for uniform motion.

Thus, the basic characteristics of nonsteady flow can differ substantially from the characteristics of uniform flow, especially when $|\delta| > 1$. In particular, when $|\delta| > 1$, the maximum value of the longitudinal velocity for accelerated flow occurs somewhere inside the flow. The possibility of shifting the velocity maximum to inside the stream for nonsteady flow has been shown experimentally [2].

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